Cheatsheet:

Axioms and Axiomatic Inference Rules Used in this Book

1 Propositional Logic

MP: Assumptions: 'p', '(p→q)'; Conclusion: 'q'

I0: '(p→p)' (**I0_SCHEMA:** '(P()→P())')

I1: '(q→(p→q))' (**I1_SCHEMA:** '(Q()→(P()→Q()))')

⋮

D: '((p→(q→r))→((p→q)→(p→r)))'

I2: '(~p→(p→q))'

N: '((~q→~p)→(p→q))'

NI: '(p→(~q→~(p→q)))'

NN: '(p→~~p)'

R: '((q→p)→((~q→p)→p))'

A: '(p→(q→(p&q)))' **NA1:** '(~q→~(p&q))' **NA2:** '(~p→~(p&q))'

O1: '(q→(p|q))' **O2:** '(p→(p|q))' **NO:** '(~p→(~q→~(p|q)))'

T: 'T' **NF:** '~F'

2 Predicate Logic

- **Modus Ponens (MP):** From ϕ and '(ϕ→ψ)', deduce ψ.
- **Universal Generalization (UG):** From ϕ deduce '$\forall x[\phi]$'.
- **Tautology:** Any formula ϕ that is a tautology.
- **Universal Instantiation (UI):** the schema '$(\forall x[\phi(x)]\rightarrow\phi(\tau))$', where $\phi(\square)$, x, and τ are (placeholders for) a parametrized formula, a variable name, and a term respectively.
- **Existential Introduction (EI):** the schema '$(\phi(\tau)\rightarrow\exists x[\phi(x)])$', where $\phi(\square)$, x, and τ are a parametrized formula, a variable name, and a term respectively.
- **Universal Simplification (US):** the schema '$(\forall x[(\psi\rightarrow\phi(x))]\rightarrow(\psi\rightarrow\forall x[\phi(x)]))$', where ψ is a (parameter-less) formula, $\phi(\square)$ is a parametrized formula, and x is a variable name. Note that the rules that define the legal instances of schemas require in particular that (the formula that is substituted for) ψ does not have (the variable name that is substituted for) x as a free variable name.

- **Existential Simplification (ES):** the schema ‘$((\forall x[(\phi(x)\rightarrow\psi)]\&\exists x[\phi(x)])\rightarrow\psi)$’, where ψ is a formula, $\phi(\Box)$ is a parametrized formula, and x is a variable name. Note once more that the rules that define the legal instances of schemas require in particular that ψ does not have x as a free variable name.
- **Reflexivity (RX):** the schema ‘$\tau{=}\tau$’, where τ is a term.
- **Meaning of Equality (ME):** the schema ‘$(\tau{=}\sigma\rightarrow(\phi(\tau)\rightarrow\phi(\sigma)))$’, where $\phi(\Box)$ is a parametrized formula, and τ and σ are terms.

2.1 Additional Axioms

In the following schemas, $\phi(\Box)$ is a parametrized formula, ψ is a formula, and x is a variable name. Note yet again that the rules that define the legal instances of schemas require in particular that ψ does not have x as a free variable name.

1. ‘${\sim}\forall x[\phi(x)]$’ is equivalent to ‘$\exists x[{\sim}\phi(x)]$’.
2. ‘${\sim}\exists x[\phi(x)]$’ is equivalent to ‘$\forall x[{\sim}\phi(x)]$’.
3. ‘$(\forall x[\phi(x)]\&\psi)$’ is equivalent to ‘$\forall x[(\phi(x)\&\psi)]$’.
4. ‘$(\exists x[\phi(x)]\&\psi)$’ is equivalent to ‘$\exists x[(\phi(x)\&\psi)]$’.
5. ‘$(\psi\&\forall x[\phi(x)])$’ is equivalent to ‘$\forall x[(\psi\&\phi(x))]$’.
6. ‘$(\psi\&\exists x[\phi(x)])$’ is equivalent to ‘$\exists x[(\psi\&\phi(x))]$’.
7. ‘$(\forall x[\phi(x)]|\psi)$’ is equivalent to ‘$\forall x[(\phi(x)|\psi)]$’.
8. ‘$(\exists x[\phi(x)]|\psi)$’ is equivalent to ‘$\exists x[(\phi(x)|\psi)]$’.
9. ‘$(\psi|\forall x[\phi(x)])$’ is equivalent to ‘$\forall x[(\psi|\phi(x))]$’.
10. ‘$(\psi|\exists x[\phi(x)])$’ is equivalent to ‘$\exists x[(\psi|\phi(x))]$’.
11. ‘$(\forall x[\phi(x)]\rightarrow\psi)$’ is equivalent to ‘$\exists x[(\phi(x)\rightarrow\psi)]$’.
12. ‘$(\exists x[\phi(x)]\rightarrow\psi)$’ is equivalent to ‘$\forall x[(\phi(x)\rightarrow\psi)]$’.
13. ‘$(\psi\rightarrow\forall x[\phi(x)])$’ is equivalent to ‘$\forall x[(\psi\rightarrow\phi(x))]$’.
14. ‘$(\psi\rightarrow\exists x[\phi(x)])$’ is equivalent to ‘$\exists x[(\psi\rightarrow\phi(x))]$’.

In the following schemas, $\phi(\Box)$ and $\psi(\Box)$ are parametrized formulas, and x and y are variable names.

15. If $\phi(x)$ and $\psi(x)$ are equivalent, then ‘$\forall x[\phi(x)]$’ and ‘$\forall y[\psi(y)]$’ are equivalent.
16. If $\phi(x)$ and $\psi(x)$ are equivalent, then ‘$\exists x[\phi(x)]$’ and ‘$\exists y[\psi(y)]$’ are equivalent.